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TIME-OPTIMAL STABILIZATION OF A PERTURBED SYSTEM WITH INVARIANT NORM

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The problem of time-optimal stabilization of a perturbed nonlinear system by controls bounded by a sphere is considered. The unperturbed system whose optimal control is determined using Schwartz's functional inequality [1, 2] is assumed to belong to the class of controlled systems with invariant norm [1]. An effective algorithm for approximate analytical derivation of Bellman's function and of perturbed optimal control based on the sufficient conditions of optimality of the dynamic programming method [3] is proposed. A procedure is developed for determining the optimal phase trajectory by successive approximations. First approximation solution of the problem of quickest braking the rotation of an almost dynamically symmetric solid body is derived with the perturbing moment of viscous friction forces taken into account [3].

1. **Statement of the problem.** Let us consider the system

$$y^{\cdot} = f_0(y) + \varepsilon f(y) + [I + \varepsilon F(y)]\, u, \quad y(0) = y_0 \tag{1.1}$$

where $y = (y_1, \ldots, y_n)$ is the phase vector whose values are contained in a bounded region which includes point $y = 0$; ε is a numberical parameter ($|\varepsilon| \leqslant \varepsilon_0, \varepsilon_0 > 0$); I is a unit matrix; f_0, and f are vectors, and F is a matrix; the dot denotes differentiation with respect to time $t \geqslant 0$, and y_0 is the initial phase state of the system. It is assumed that u is the control vector function of dimension $n \geqslant 1$ which satisfies the constraint $|u|^2 \leqslant u_0^2$, $u_0 = \text{const}$. It can be assumed without loss of generality that $u_0 = 1$ and, also, that functions f_0, f and F have a reasonable number of derivatives with respect to y in the indicated region. The dependence of these functions on parameter ε can be continuous, but is not defined here.

It is assumed that the uncontrolled unperturbed system, i.e. system (1.1), when $u \equiv 0$, and $\varepsilon = 0$ has the invariant norm [1]

$$\eta' f_0(y) \equiv 0, \ \eta = y h^{-1}, \quad h = |y|, \ h \in [0, h_0], \ h_0 = |y_0| \tag{1.2}$$

where η is a unit vector (column vector) directed along vector y, and η' is a transposed vector. It follows from (1.2) that $h(t) = h_0 = \text{const}$, since $h^{2\cdot} = 0$ and, consequently, $|y_i(t)| \leqslant h_0$. Note that in mechanics forces f_0 defined by (1.2) are called gyroscopic forces [5] whose power at any instant of time is zero.

The design of a time-optimal control which would bring the unperturbed system

(1.1) to the coordinate origin is determined by the inequality $|\eta' u| \leqslant 1$ (the Schwartz inequality [1, 2])

$$u_0^*(y) = -\eta, \quad u_0^*[t] = -\eta_0(t) \equiv -y_0^*(t, y_0) / h_0^*(t, h_0) \tag{1.3}$$
$$h_0^* = h_0(1 - t / T_0^*), \quad T_0^* = h_0, \quad u_0^*[t] = u_0^*(y_0^*(t, y_0))$$

where T_0^* is the time of response, $y_0^*(t, y_0)$ is the unpreturbed phase trajectory, and $y_0^*(T_0^*, h_0) = 0$. In this problem the Bellman function, i.e. the positive smooth solution of the related Cauchy problem for the Hamilton-Jacobi equation [1] (Bellman equations [3]), is $T_0(y) = h$ (see Sectn. 3). Note that the use of control (1.3) for stabilizing the perturbed system (1.1) generally results in an error $O(\varepsilon)$ relative to the phase trajectory and to the functional of the time of response T. This statement follows from the equation

$$h^{\cdot} = -1 + \varepsilon\eta'(f - F\eta), \quad h(0) = h_0 \tag{1.4}$$

and from the boundedness of the multiplier at ε in (1.4).

In applied problems it is often necessary to obtain a more accurate solution of the problem of optimal stabilization of a perturbed system, taking into account parameter ε. We have the following problem. Determine the time-optimal control law $u = u(y, \varepsilon)$, the minimum time $T = T(y_0, \varepsilon)$, and the perturbed phase trajectory $y = y(t, y_0, \varepsilon)$ with $(y(0, y_0, \varepsilon) = y_0, y(T, y_0, \varepsilon) = 0)$ and specified accuracy with respect to the small parameter ε.

Problems of optimal motion control of systems in a similar formulations were investigated in [2, 4, 6-8] by the method of perturbations.

2. **Controlled rotations of a solid body.** As an example of a controlled unperturbed system with invarient norm we consider the system of Euler's dynamic equations [1, 9, 10]

$$I_1\omega_1^{\cdot} + (I_3 - I_2)\omega_2\omega_3 = M_1, \quad \omega_1(0) = \omega_{10} \qquad (1, 2, 3) \tag{2.1}$$

where $M_i = b_i u_i$ $(i = 1, 2, 3)$ are the controlling moments $b_i = \text{const} > 0$, and u_i are controls bounded by the inequality $u_1^2 + u_2^2 + u_3^2 \leqslant 1$. The problem is to bring the phase point of the system from the initial state $\omega_i(0) = \omega_{i0}$ to the coordinate origin $\omega_i(T) = 0$ in the shortest time T, in other words, to determine the time-optimal stabilization of system (2.1).

Introducing variables $z_i = L_i b_i^{-1}$, where $L_i = I_i\omega_i$ are components of the angular momentum vector defined in coupled axes, we reduce system (2.1) to the form of the unperturbed equations (1.1)

$$z_1^{\cdot} + (I_3 - I_2)I_2^{-1}I_3^{-1}b_2b_3b_1^{-1}z_2z_3 = u_1, \; z_1(0) = L_{10}b_1^{-1} \; (1, 2, 3) \tag{2.2}$$

The invariance condition (1.2) for system (2.2) is satisfied if parameters I_i, and b_i satisfy the relationship

$$I_1(I_3 - I_2)b_2^2b_3^2 + I_2(I_1 - I_3)b_1^2b_3^2 + I_3(I_2 - I_1)b_1^2b_2^2 = 0 \quad (2.3)$$

The set of such parameters is nonempty. Let $I_3 \geqslant I_2 \geqslant I_1 > 0$. Then for $I_3 > I_1$ equality (2.3) can be represented in the form

$$b_1^2b_2^{-2} = I_1(I_3 - I_2)b_3^2b_2^{-2}[I_2(I_3 - I_1)b_3^2b_2^{-2} - I_3(I_2 - I_1)]^{-1}$$

This formula is meaningful if b_3^2 is fairly large, i.e. when for a given b_2^2 the inequality $b_3^2 > b_2^2I_3I_2^{-1}(I_2 - I_1)\times(I_3 - I_1)^{-1}$ is satisfied; the same holds for b_1^2.

Let us consider some particular cases in which the invariance condition is satisfied.

1) In the case of a solid body with arbitrary moments of inertia $I_3 \geqslant I_2 \geqslant I_1$ equality (2.3) is satisfied when: a) $b_1 = b_2 = b_3 = b$ (see [1,4,8,10]); then vector z_i $(i = 1, 2, 3)$ and the moment of momentum vector $L_i : L_i = bz_i$; are collinear; and b) when $b_1 = bI_1\sqrt{I_2I_3}$, $z_1 = \omega_1 / (b\sqrt{I_2I_3})$ with $b_1 \leqslant b_2 \leqslant b_3 \cdot 1,2,3)$

2) For a dynamically symmetric body $I_1 = I_2 = I_0$ equality (2.3) is satisfied for $b_1 = b_2 = b$; I_3 and b_3 are arbitrary $(I_3 \leqslant 2I_0)$, and when $I_0b^{-1} = I_3b_3^{-1}$ vector z_i and the angular velocity vector $\omega_i : z_i = I_0b^{-1}\omega_i$ are collinear (see [4,8]).

3) In the case of a spherically symmetric solid body $(I_1 = I_2 = I_3 = I_0)$, formula (2.3) is valid for any b_1, b_2, b_3 [4,8].

Thus when condition (2.3) is satisfied, the optimal braking of solid body rotation is defined by an expression of the form (1.3)

$$u_i^* = -z_iz^{-1}, \quad z = (z_1^2 + z_2^2 + z_3^2)^{1/2}, \quad (2.4)$$

$$T_0^* = z_0, \quad z_0^* = z_0(1 - t / T_0^*)$$

The optimal trajectory $z_i(t)$ can be derived from the known formulas for uncontrolled motion of a solid body [4,8,9,10]. In the general case 1) the free rotation is defined by elliptic equations [9]. In fact, if ω_i° is the general solution of system (2.1) with $M_i \equiv 0$ then

$$L_i^\circ = I_i\omega_i^\circ, \quad \omega_i^\circ = \omega_i^\circ(t + \tau, L^\circ, E^\circ) \quad (2.5)$$

where ω_i° is a 2π -periodic function of phase $\psi = \Omega(t + \tau)$, and the frequency Ω depends on constants of the modulus of the moment of momentum L° and energy E°. These parameters and the phase constant τ are determined by the initial conditions (2.1).

Solution for the controlled system (2.1) when $u_i = u_i^*$ (see (2.5)) is of the form $L_i = zl_i$, where functions l_i satisfy the system

$$\frac{dl_1}{ds} + \frac{I_3 - I_2}{I_2 I_3} l_2 l_3 = 0, \quad l_1(0) = \frac{L_{10}}{z_0} \quad (1,\ 2,\ 3)$$

$$s = \int_0^t z\, dt' = z_0 t \left(1 - \frac{t}{2T_0^*}\right)$$

As the result, on the basis of (2.5) we obtain for the optimal trajectory the formula.

$$L_i = zL_i^\circ (s + \theta,\ L^\circ z_0^{-1},\ E^\circ z_0^{-2}) \qquad (i = 1,\ 2,\ 3),\ \theta = \text{const} \tag{2.6}$$

Note that all components of the angular momentum vector (2.6), as well as the rate of variation of phase ψ, simultaneously vanish when $z = 0$, i.e. at the instant of time $t = T_0^*$ [4, 8, 10].

3. **D e r i v a t i o n o f t h e a p p r o x i m a t e o p t i m a l c o n t r o l b y t h e m e t h o d o f d y n a m i c p r o g r a m m i n g.** Solution of the problem of defining the time-optimal stabilization of the perturbed system (1.1) consists of finding a nonnegative differentiable Bellman function $T\ (y,\ \varepsilon)$ which satisfies the Hamilton–Jacobi equation [1] (the Bellman equation [3]) with boundary condition

$$\frac{\partial T}{\partial y} [f_0(y) + \varepsilon f(y)] + \min_{u^2 \leqslant 1} \left\{ \frac{\partial T}{\partial y} [I + \varepsilon F(y)]\, u \right\} = -1, \quad T(0, \varepsilon) \equiv 0 \tag{3.1}$$

where $\partial T\ /\ \partial y$ is a column vector, and the expression $\partial T\ /\ \partial y\ , f_0$ and similar are scalar products. The minimization of expressions in braces in (3.1) yields the closed Cauchy problem

$$\frac{\partial T}{\partial y} [f_0(y) + \varepsilon f(y)] - \left| \frac{\partial T}{\partial y} [I + \varepsilon F(y)] \right| = -1, \quad T(0, \varepsilon) \equiv 0 \tag{3.2}$$

and the formula for optimal control

$$u^* = -(I + \varepsilon F') \left(\frac{\partial T}{\partial y}\right)' \left| \frac{\partial T}{\partial y} (I + \varepsilon F) \right|^{-1} \tag{3.3}$$

The solution of problem (3.2) is expressed in the form of expansion

$$T\ (y,\ \varepsilon) = T_0\ (y) + \varepsilon T_1\ (y) + \ldots + \varepsilon^j T_j\ (y) + \ldots, \tag{3.4}$$
$$T_j\ (0) = 0, \quad j = 0, 1, \ldots$$

where the unknown coefficients T_j and functions f and F(see Sectn. 1) may continuously depend on parameter ε. The form of their dependence is not indicated. Functions $T_j(y)$ are obtained by successive solution of the coupled system of equations in partial derivatives [4, 6].

$$\frac{\partial T_0}{\partial y} f_0(y) - \left|\frac{\partial T_0}{\partial y}\right| = -1, \quad T_0(0) = 0 \tag{3.5}$$
$$\frac{\partial T_j}{\partial y} f_0(y) - \frac{\partial T_j}{\partial y}\left(\frac{\partial T_0}{\partial y}\right)' \left|\frac{\partial T_0}{\partial y}\right|^{-1} = V_j(y), \quad T_j(0) = 0, \quad j \geqslant 1$$

Functions V_j in (3.5) are determined by solving preceding equations, since at any j-th step they are calculated in terms of functions obtained in preceding steps, i.e. f, F and $\partial T_0 / \partial y$, $\partial T_1 / \partial y, \ldots, \partial T_{j-1} / \partial y$, e.g. for $j = 1$

$$V_1(y) = \frac{\partial T_0}{\partial y} F' \left(\frac{\partial T_0}{\partial y}\right)' \left|\frac{\partial T_0}{\partial y}\right|^{-1} - \frac{\partial T_0}{\partial y} f \tag{3.6}$$

For any arbitrary subscripts $j = 1, 2, \ldots$ functions $V_j(y)$ are determined by formulas

$$V_j = W_j - \frac{\partial T_{j-1}}{\partial y} f, \quad W_j = W_j\left(F, \frac{\partial T_0}{\partial y}, \ldots, \frac{\partial T_{j-1}}{\partial y}\right) \tag{3.7}$$
$$\left|\frac{\partial T}{\partial y}(I + \varepsilon F)\right| = \left[\frac{\partial T}{\partial y}(I + \varepsilon F)(I + \varepsilon F')\left(\frac{\partial T}{\partial y}\right)'\right]^{1/2} =$$
$$\left|\frac{\partial T_0}{\partial y}\right| + \varepsilon\left[\frac{\partial T_1}{\partial y}\left(\frac{\partial T_0}{\partial y}\right)' \left|\frac{\partial T_0}{\partial y}\right|^{-1} + W_1\right] + \ldots +$$
$$\varepsilon^j\left[\frac{\partial T_j}{\partial y}\left(\frac{\partial T_0}{\partial y}\right)' \left|\frac{\partial T_0}{\partial y}\right|^{-1} + W_j\right] + \varepsilon^{j+1} \ldots$$

From formulas (1.3) and the definition of Bellman's function T follows that $T_0(y) = h = |y|$ is the solution of the first Cauchy problem (3.5), i.e. T_0 the Bellman function of the unperturbed problem of time-optimal stabilization. Since $|\partial T_0 / \partial y| = 1$, hence the formal expansions (3.4)-(3.7) obtained above are valid for fairly small values of parameter ε. Function T_0 defines optimal control in the form of u^* in (3.3) with an error $O(\varepsilon)$: $u_0^* = -\eta$ (see 1.4)).

Solutions for the Cauchy problem (3.5) for $j \geqslant 1$ are obtained successively, as in [4, 8], using the method of characteristics [11, 12]. The equations of characteristics reduce to the form

$$\frac{dy_1}{f_{01}(y) - \eta_1} = \ldots = \frac{dy_n}{f_{0n}(y) - \eta_n} = \frac{dT_j}{V_j(y)} = \frac{dh}{-1} \tag{3.8}$$

Let the known function y^* be the general solution of system (3.8) of the form (c is the general integral)

$$y^* = y^*(h, y_0), \quad y^*(h_0, y_0) = y_0, \quad |y^*(h_0, y_0)| = h_0 \qquad (3.9)$$
$$(c = C(y), \quad c' = (c_1, \ldots, c_{n-1}))$$

The sought solutions are then determined by squaring

$$T_j(y) = -\int_0^h V_j(y^*(l, y^*(h, y)))\,dl, \quad h = |y|, \quad j = 1, 2, \ldots \qquad (3.10)$$
$$(y^*(h, y) \equiv y)$$

Thus for determining coefficients T_j in (3.10) of expansion (3.4) it is necessary to be able to derive the general solution of the unperturbed controlled system (1.1) with $u = u_0^* = -\eta$ (see Sects. 2, 4, 5).

To prove the method of deriving the approximate solution of Eq. (3.2) with specified boundary condition, developed in Sectn. 3, it is necessary to consider the problem of bringing the phase point of system (1.1) to the μ -neighborhood of the coordinate origin $y = 0$ in the shortest time and, then, pass to the limit $\mu \to 0$. This yields the expressions of coefficients of expansion (3.4) that coincide with those in (3.10).

The vector function of optimal control $u^*(y, \varepsilon)$ can be expressed in the form of expansion similar to (3.4)

$$u^*(y, \varepsilon) = -\eta + \varepsilon u_1^* + \ldots + \varepsilon^j u_j^* + \varepsilon^{j+1} \ldots \equiv \qquad (3.11)$$
$$-\eta + \varepsilon u_{(1)}^*(y, \varepsilon)$$

where coefficients $u_j^*(y)$ are determined in terms of derivatives of functions $T_j(y)$ after the substitution of (3.4) into (3.3) into (3.3), and equating coefficients at like powers of ε. For $j = 1$ we have (see (3.7)).

$$u_1^{*\prime}(y) = \eta'\left(\frac{\partial T_1}{\partial y}\eta - \eta\frac{\partial T_1}{\partial y}\right) + U_1'(y), \quad U_1' = \eta'(W_1 - F) \qquad (3.12)$$

For any arbitrary $j \geqslant 1$ functions $u_j^*(y)$ are

$$u_j^{*\prime}(y) = \eta'\left(\frac{\partial T_j}{\partial y}\eta - \eta\frac{\partial T_j}{\partial y}\right) + U_j'(y) \qquad (3.13)$$

where functions $U_j(y)$ are defined similarly to functions W_j in (3.7) and depend on

$$F, \ \partial T_0 / \partial y = \eta', \ \partial T_1 / \partial y, \ldots, \partial T_{j-1} / \partial y.$$

R e m a r k. If the analytic form of the generating solution (3.9) is not known, the algorithm of the approximate calculation of Bellman's function can be obtained as follows. Coefficients T_j in (3.10) are determined for the fairly dense set of points y_k: $y_{i1}, \ldots, y_{ik_i}, \ldots, y_{iN_i} = h_0$ where the first subscript $i = 1, 2, \ldots, n$ is the ordinal number of the component of vector y, the second subscript k_i denotes the ordinal number of the separation point, and N_i is the number of separation points of the interval of the i-th component variation. Then we determine function y_* which is the set of solutions of the Cauchy problem for the unperturbed controlled system.

$$y^{\cdot} = f_0(y) - \eta, \ y(0) = y_k \quad (k = (k_1, \ldots, k_n), \ 1 \leqslant k_i \leqslant N_i) \qquad (3.14)$$

Numerical integration determines also that set in the discrete set of points of argument $t \in [0, h_k]$

$$y_{lk} = y_*(t_l, y_k), \qquad l = 1, \ldots, N(k), \ h_k = |y_k|$$

Since $h_l = h_k - t_l$, the sought set of functions $y^*(h, y)$ in (3.9) specified in the discrete set of points h_l is of the form

$$y_{lk} = y_*(h_k - h_l, y_k) \equiv y^*(h_l, y_k), \quad 0 \leqslant h_l \leqslant h_k$$

Finally, we integrate in conformity with (3.10) functions $V_j(y^*(h_l, y_k))$ of the discrete argument h_l which depends on the discrete vector parameter y_k, and obtain coefficients $T_j(y_k)$ specified in a reasonably dense set of points $|y_k| \leqslant h_0$. The integration may be carried out by the method of rectangles or by some other more accurate scheme. The approximate calculation of controls $u^*(y, \varepsilon)$ (3.11) by formulas (3.12) and (3.13) may be carried out by finite difference differentiation or some other means appropriate for solving the system in variations for problem (3.14).

A similar algorithm of approximate determination based on dynamic programming method can be formulated for discrete systems of optimal control most suitable for computer calculations. However that problem requires separate consideration.

4. Derivation of the approximate optimal trajectory. The substitution of the optimal control $u^*(y, \varepsilon)$ (3.11) into (1.1) yields the closed Cauchy problem for the determination of optimal phase trajectories $y = y(t, y_0, \varepsilon)$, which may be represented in the form of expansions or successive approximations in powers of parameter ε. Let us assume that the general solution (1.3) of the unperturbed controlled system (1.1) (in particular, in the case of system (2.1) this is function L_i (2.6)) or the complete system of integrals of the (3.9) type are known. The perturbed optimal trajectory or the osculating variables (integrals) c and the "phase" ψ can be determined in the form of quadratures with an accuracy with respect to ε equal to that with which the vector function of control $u^*(y, \varepsilon)$ was calculated.

Thus the solution of the [problem of] unperturbed controlled system $y_0^*(t, y_0)$, $y_0^*(0, y_0) = y_0$ is known. Solution of the perturbed system (1.1) is formulated as $y = y_0{}^* + \varepsilon x\,(t, \varepsilon)$. The unknown vector x is obtained as the solution of the Cauchy problem

$$x^{\cdot} = \left(\frac{\partial f_0}{\partial y_0{}^*} h_0{}^* - I + \eta_0\eta_0{}'\right)\frac{x}{h_0{}^*} + f^* - F^*\eta_0 + \overset{*}{u}_{(1)} + \varepsilon P\,(t, x) \tag{4.1}$$
$$f^*(t) \equiv f(y_0{}^*), \quad F^*(t) \equiv F(y_0{}^*), \ldots, \eta_0(t) = y_0{}^*/h_0{}^*, \quad x(0, \varepsilon) \equiv 0$$

where, as previously, the dependence of functions $f, F, u_{(1)}{}^*$, and P is not defined; function P is known with the required degree of accuracy. Solution of the quasi-linear system (4. 1) is obtained by successive approximations using the scheme [12]

$$x_k(t, \varepsilon) = x_0(t) + \varepsilon X(t) \int_0^t X^{-1}(t')\,P\,(t', x_{k-1}(t', \varepsilon))\,dt' \tag{4.2}$$
$$x_0(t) = X(t) \int_0^t X^{-1}(t')(f^* - \overset{*}{F}\eta_0 + \overset{*}{u}_{(1)})\,dt', \quad k = 1, 2, \ldots, j-1$$

where
$X\ (t) = \partial y_0{}^*/\partial y_0$ is the known fundamental matrix of solutions of the unperturbed system (4. 1). The successive approximations x_k (4. 2) determine for fairly small $|\varepsilon|$ the unique solution of system (4. 1) $x^*\ (t, \varepsilon)$ with an error $O\ (\varepsilon^j)$, and $y^*\ (t, \varepsilon) = y_0{}^*\ (t, y_0) + \varepsilon x^*\ (t, \varepsilon)$ is the system perturbed optimal trajectory determined with the same error $O\ (\varepsilon^{j+1})$ as that of the control function $u^*\ (y, \varepsilon)$ (3. 11).

The solution of the problem of determining perturbed integrals of the (3. 9) type defined by the equations

$$c^{\cdot} = \varepsilon\frac{\partial C}{\partial y^\circ}\varphi(y^\circ), \quad \psi^{\cdot} = 1 + \varepsilon\frac{\partial \sigma}{\partial y^\circ}\varphi(y^\circ), \quad c(0) = C(y_0), \quad \psi(0) = \sigma(y_0) \tag{4.3}$$
$$c = C\ (y), \quad \psi = \sigma\ (y), \quad y = y^\circ\ (c, \psi), \quad \varphi = f + Fu^* + u_{(1)}{}^*$$

where
$\psi = \sigma\ (y) = t + \tau$ is a time dependent integral, reduces to quadratures similar to (4. 2).

The approximate solution of system (4. 3) is obtained by successive approximations by a scheme of the type of (4. 2) [12]

$$c_k = C\,(y_0) + \varepsilon\int_0^t \frac{\partial C_{k-1}}{\partial y^\circ}\,\varphi\,(y^\circ(c_{k-1}, \psi_{k-1}))\,dt', \quad c_0 = C\,(y_0) \tag{4.4}$$
$$\psi_k = t + \sigma\,(y_0) + \varepsilon\int_0^t \frac{\partial \sigma_{k-1}}{\partial y^\circ}\,\varphi\,(y^\circ(c_{k-1}, \psi_{k-1}))\,dt', \quad \psi_0 = t + \sigma\,(y_0)$$

Thus the constructions derived above reduce to quadratures when the general solution $y_0{}^*\ (t, y_0)$ for the unperturbed controlled system (1. 1) is known. It should be noted that (as shown in Sect. 2 for Euler's equation (2. 1) in certain important applications the general solution of the controlled system

$$y^{\cdot} = f_0(y) - \eta, \quad y(0) = y_0 \tag{4.5}$$

can be obtained on the basis of the known uncontrolled motion $v = v(t, a, h_0)$, that satisfies the system with invariant norm [1]

$$v^{\cdot} = f_0(v), \quad v(0) = y_0, \quad a = (a_1, \ldots, a_{n-1}), \quad |v| = h_0$$

In fact (see Sectn. 2) the substitution $y = h_0{}^* w$, where w is an unknown vector function, reduces system (4.5) to the form

$$w^{\cdot} = f_0(h_0{}^* w) / h_0{}^*, \quad w(0) = y_0 h_0{}^{-1}, \quad |w| = 1 \tag{4.6}$$

Let $f_0(y)$ be a homogeneous function of power $m \geqslant 1$ of y i.e. $f_0(\gamma y) = \gamma^m f_0(y)$. The system of Eqs. (4.6) then assumes the form

$$\frac{dw}{ds} = f_0(w), \quad s = \frac{1}{m}(h_0^m - h_0^{*m}), \quad w\big|_{s=0} = y_0 h_0^{-1}$$

As the result, we have for $y_0{}^*$ the expression

$$y_0{}^*(t, y_0) = h_0{}^*(t, h_0) v(s, a, 1), \quad v(0, a, 1) = y_0 h_0{}^{-1} \tag{4.7}$$

where v is by assumption a known function that defines the uncontrolled motion. The particular case of a system with invariant norm for which $m = 2$ was considered in Sect. 2.

5. Braking of rotation of a solid body with allowance for perturbing moments. The approximate solution of the problem of time-optimal stabilization of a solid body almost dynamically symmetric is investigated with allowance for the perturbing moment of viscous friction forces [4, 8]. It is assumed that the parameters of system (2.1) are

$$\begin{gathered} I_{1,2} = I_0(1 + \varepsilon\varkappa_{1,2}), \quad b_{1,2} = b_0(1 + \varepsilon\beta_{1,2}) \\ M_i = b_i u_i - \varepsilon \sum_{j=1}^{3} \Lambda_{ij}\omega_j \quad (i = 1, 2, 3), \quad I_3 \neq I_0 \end{gathered} \tag{5.1}$$

where ε ($|\varepsilon| \ll 1$) is a small numerical parameter, $\varkappa_1$, $\varkappa_2$, β_1, and β_2 are constant numbers of order unity, and $(\varepsilon\Lambda_{ij})$ is the tensor of the perturbing moment of viscous friction forces (a nonnegative definite constant matrix). When $\varepsilon = 0$ the solution of the problem of time-optimal braking is

$$u_i^* = -x_i h^{-1}, \quad T_0(x) = h = |x| \tag{5.2}$$
$$x_i = I_i \omega_i b_i^{-1}, \quad h_0^* = h_0 (1 - t / T_0^*), \quad T_0^* = h_0, \; h_0 = |x_0|$$

The optimal phase trajectory is of the form (see Sect. 2 and (4.7)

$$x_1 = h_0^* v_1 = (1 - t | T_0^*) |x_{0\perp}| \cos(s^\circ + \tau), \quad \cos\tau = x_{10} |x_{0\perp}|^{-1} \tag{5.3}$$
$$x_2 = h_0^* v_2 = (1 - t / T_0^*) |x_{0\perp}| \sin(s^\circ + \tau), \; \sin\tau = x_{20} |x_{0\perp}|^{-1}$$
$$x_3 = h_0^* v_3 = (1 - t / T_0^*) x_{30}, \quad s^\circ = d x_{30} t (1 - t / (2T_0^*))$$
$$d = (I_3 - I_0) b_3 / (I_0 I_3)$$

The optimal control expressed as a function of time is obtained from formulas (5.2) an d (5.3)

$$u_1^*[t] = -|x_{0\perp}| h_0^{-1} \cos(s^\circ + \tau), \quad u_2^*[t] = \tag{5.4}$$
$$-|x_{0\perp}| h_0^{-1} \sin(s^\circ + \tau), \; u_3^*[t] = -x_{30} h_0^{-1}$$

When $\varepsilon \neq 0$ system (2.1), (5.1) reduces to the form (1.1) [4,8]

$$x_1^\cdot + d x_2 x_3 = u_1 + \varepsilon\varphi_1(x) + \varepsilon a_1 x_2 x_3, \quad x_{10} = I_1 \omega_{10} b_1^{-1} \tag{5.5}$$
$$x_2^\cdot - d x_1 x_3 = u_2 + \varepsilon\varphi_2(x) + \varepsilon a_2 x_1 x_3, \; x_{20} = I_2 \omega_{20} b_2^{-1}$$
$$x_3^\cdot = u_3 + \varepsilon\varphi_3(x) + \varepsilon a_3 x_1 x_2, \quad x_{30} = I_3 \omega_{30} b_3^{-1}$$

where φ_i are transformed components of the perturbing moment of viscous friction forces (without the ε multiplier) defined by

$$\varphi_i(x) = -\sum_{j=1}^{3} \lambda_{ij} x_j, \quad \lambda_{ij} = \Lambda_{ij} I_j^{-1} b_i^{-1} \quad (i, j = 1, 2, 3)$$

The constants a_i are determined by formulas

$$\varepsilon a_1 = d - \frac{I_3 - I_2}{I_2 I_3} \frac{b_2 b_3}{b_1}, \quad \varepsilon a_2 = \frac{I_3 - I_1}{I_1 I_3} \frac{b_1 b_3}{b_2} \quad \varepsilon a_3 = -\frac{I_2 - I_1}{I_1 I_2} \frac{b_1 b_2}{b_3}$$

Let us derive the solution of the problem of the first approximation optimal control $T(x, \varepsilon) = h + \varepsilon T_1 + \varepsilon^2 \ldots$ In conformity with (3.6) function $V_1(x)$ which appears in the definition of coefficient $T_1(x)$ (3.10), can be decomposed into two terms $V_1 = V_{1G} + V_{1F}$ that define perturbations induced by gyroscopic moments and friction forces, respectively,

$$V_{1G}(x) = \frac{x_1x_2x_3}{h}\sum_{i=1}^{3} a_i, \quad V_{1F}(x) = \frac{1}{h}\sum_{i,\,j=1}^{3}\lambda_{ij}x_ix_j \tag{5.6}$$

As the result, for T_{1G} $(T_1 = T_{1G} + T_{1F})$ we obtain [4, 8]

$$T_{1G}(x) = -{}^1/_2(a_1 + a_2 + a_3)\,x_3h^{-3}\{[(x_2^2 - x_1^2)\cos\theta + 2x_1x_2\sin\theta]\int_0^h l^2\sin\beta l^2\,dl + [2x_1x_2\cos\theta - (x_2^2 - x_1^2)\sin\theta]\int_0^h l^2\cos\beta l^2 dl\}, \quad \theta = dx_3h, \quad \beta = dx_3h^{-1} \tag{5.7}$$

For $x_{30} \neq 0$ this formula reduces to Fresnel integrals [13]. Formula (5.7) is derived using the expression for solutions of the form (3.9)

$$x_1 = h\,(\eta_{10}\cos s_0 - \eta_{20}\sin s_0), \quad x_2 = h\,(\eta_{10}\sin s_0 + \eta_{20}\cos s_0) \tag{5.8}$$
$$x_3 = h\eta_{30} \quad (s_0 = {}^1/_2dh^2\eta_{30}, \quad \eta_0 = x_0h_0^{-1})$$

using (5.6) and substituting (5.8) for T_{1F} in (3.10) we obtain the final formula

$$T_{1F}(x) = \sum_{i,\,j=1}^{3}\lambda_{ij}\alpha_{ij}(x, \eta_0), \quad \alpha_{ij} = \alpha_{ji} = \int_0^h x_i(l, \eta_0)\,x_j(l, \eta_0)\frac{dl}{l} \tag{5.9}$$

After integration we substitute in (5.9) for components of vector η_0, their expressions in (5.8)

$$\eta_{10} = \eta_1\cos s + \eta_2\sin s, \quad \eta_{20} = \eta_2\cos s - \eta_1\sin s \tag{5.10}$$
$$\eta_{30} = \eta_3, \quad s = {}^1/_2dh^2\eta_3$$

The coefficients α_{ij} (x, η_0) in (5.9) are explicitly obtained, e.g.

$$\alpha_{11} = (\eta_{10}^2 + \eta_{20}^2)\frac{h^2}{4} + \frac{\eta_{10}^2 - \eta_{20}^2}{4d\eta_{30}}\sin 2s_0 - \frac{\eta_{10}\eta_{20}}{2d\eta_{30}}(1 - \cos 2s_0) \tag{5.11}$$

If η_{30} is small ($|\eta_{30}| \ll 1$), formulas (5.7) and (5.11) in linear and cubic approximations with respect to η_{30} and h, respectively, are considerably simplified

$$T_{1G} = -\frac{x_1x_2x_3}{3}\sum_{i=1}^{3} a_i + O(\eta_{30}^2), \quad T_{1F} = -\frac{1}{2}\sum_{ij=1}^{3} \lambda_{ij}x_ix_j + O(\eta_{30}h^4) \quad (5.12)$$

Then, using the known expression for the coefficient $T_1(x)$ we determine by formula (3.12) the vector function u_1^* which defines the optimal control $u^* = -\eta + \varepsilon u_1^*$ in the first approximation with respect to ε, i.e. with an error $O(\varepsilon^2)$ with respect to the functional and phase trajectory. The first approximation phase trajectory is determined by the quadratures (4.2) using the known general solution (5.3) for x_i of the unperturbed system (5.5). Note that the moment of friction forces reduces the time of response T [4, 8].

The procedures developed here make it possible to solve approximately in quadratures problems of time-optimal stabilization of perturbed systems of the form (1.1), (1.2). Their application requires the ability to formulate the general solution of problems of unperturbed controlled system with invariant norm, although in a number of important applications it is sufficient to know only the uncontrolled motion.

Note that proposed approach makes possible the solution of the problem of designing [optimal controls] for systems of a more general form than (1.1), such, as for instance,

$$x^{\cdot} = f_0(t, x) + \varepsilon f(t, x) + b(t, h)\,[S(t, x) + \varepsilon F(t, x)]\,u$$
$$x(t_0) = x_0, \quad h = |x|, \; |u| \leqslant 1$$

where f_0, and f are vector functions, b is a scalar function, and S is an orthogonal matrix. All functions are assumed to be reasonably smooth in the considered region of argument variation. A more general assumption is made about function f_0 than specified by (1.2), namely, $\eta' f_0(t, x) = \varphi(t, h)$ (see [1, 8]). In the case of an unperturbed controlled system with invariant norm $\varphi \equiv 0$.

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